Class 13, given on Feb 1, 2010, for Math 13, Winter 2010

1. A spherical coordinate example

We will not spend too much time on spherical coordinates, but let us look at one more basic example to see spherical coordinates in action.

Example. Consider a solid, of uniform density, which fills the space E between $x^2+y^2+z^2 \le 4, x^2+y^2+z^2 \le 9$, and also has $z \ge 0$. What is the center of mass of this solid?

Symmetry immediately indicates that $\overline{x} = \overline{y} = 0$. To calculate $\overline{z}$, we first should calculate the mass of this solid. Let us assume that $\rho(x,y,z) = 1$; then the mass is the volume of the solid, which is equal to

$$\frac{4}{3}\pi(3^3 - 2^3)\frac{1}{2} = \frac{38\pi}{3}.$$

The moment about the xy plane is equal to

$$\iiint_E z\, dV.$$

The region E is described by spherical coordinates $2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2$. Therefore, this integral is equal to

$$\int_0^{\pi/2}\int_0^{2\pi}\int_2 3(\rho\cos\phi)(\rho^2\sin\phi)\, d\rho\, d\theta\, d\phi.$$

We use the trick which tells us that this integral is equal to the product of three single integrals, since the integrand is a product of r^3 with $\cos\phi\sin\phi$ and 1:

$$\left(\int_0^{\pi/2}\cos\phi\sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_2^3 \rho^3\, d\rho\right) = \left(\frac{\sin^2\phi}{2}\Big|_0^{\pi/2}\right)2\pi\left(\frac{\rho^4}{4}\Big|_2^3\right) = \frac{65\pi}{4}.$$

Therefore, $\overline{z}$ is given by

$$\frac{65\pi}{4}\cdot\frac{3}{38\pi} = \frac{195}{152}.$$

2. Change of variables: the Jacobian

So far, we have seen three examples of situations where we 'change variables' to help us evaluate integrals: when we change from rectangular coordinates in $\mathbb{R}^2$ to polar coordinates, when we change from rectangular in $\mathbb{R}^3$ to cylindrical coordinates, and when we change from rectangular to spherical coordinates. These are all special instances of *change of variables*, where we replace variables in an integral with other variables, according to how they are related to each other. In each case, there was a formula which connected an integral in one coordinate system to an integral in the other coordinate system, and in each case an extra factor appeared in the integrand. We want to give a brief idea of just why these factors appear, although a full explanation requires linear algebra.

Let us take a step back and think about just how we change variables before we actually think about integration. In each situation, we're given two or three variables, such as x, y, z, and two or three other variables, such as r, θ, z, or ρ, θ, ϕ, which we express x, y, z in terms of. Let's look at the case of polar coordinates, where $x = r\cos\theta, y = r\sin\theta$. We can encapsulate the relationship between x, y, r, θ by defining a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ which takes (r, θ) to $(r\cos\theta, r\sin\theta)$. In other words, T takes the polar coordinates of some points to the rectangular coordinates of that point.

In particular, suppose we want to integrate a function $f(x, y)$ over a region R using polar coordinates. Then we start by finding a region S which describes R in polar coordinates: in other words, we want $T(S) = R$. Furthermore, we want T to be a *one-to-one* map on S; ie, we only want one point in S to map to a given point in R. This manifests itself in the requirement that $r \geq 0, 0 \leq \beta - \alpha \leq 2\pi$ when we integrate over a polar rectangle, for example. Strictly speaking, we can slightly relax the one-to-one condition, and possibly allow T to not be one-to-one for points on the boundary of S. In any case, we end up with a formula which describes the integral of $f(x, y)$ over R as an integral of $f(T(r, \theta)) = f(r\cos\theta, r\sin\theta)$ over S.

This new terminology brings out the essential features when changing variables. In the case where we are dealing with two variables, we have a function T, defined on points (u, v), which sends (u, v) to (x, y), where $x = g(u, v), y = h(u, v)$, are some functions of x, y. If S is a region in the uv plane on which T is a one-to-one function, except possibly at the boundary of S, and $R = T(S)$, then we want an expression which relates an integral using xy coordinates to one using uv coordinates.

For technical reasons, we require that T be a C^1 transformation, which is a fancy way of saying that $g(u, v), h(u, v)$ should have continuous first order partial derivatives. How are integrals using uv coordinates related to integrals using xy coordinates? Recall that when defining an integral, we used Riemann sums, whose individual terms looked like $f(x^*, y^*)\Delta x\Delta y$, where x^*, y^* was a point inside a box of dimensions $\Delta x \times \Delta y$.

If we are integrating over S in the uv plane, a typical term in a Riemann sum will be over a rectangle with side lengths $\Delta u, \Delta v$. Suppose the rectangle this term represents has a vertex (u, v) in the lower left hand corner. This is a small rectangle in the uv plane. What is the image of this rectangle under the transformation T? Well, it might be hard to determine exactly what this image is (T might well be a non-linear map), but we can approximate the image with a parallelogram by pretending that T is linear. In particular, the way we pretend that T is linear is by looking at partial derivatives of g, h.

For example, suppose we want to approximate the image of $(u + \Delta u, v)$ under T. Then we are incrementing the u-coordinate by a small amount Δu. The x coordinate of $T(u, v)$ is $g(u, v)$. The x coordinate of $T(u + \Delta u, v)$ is given by $g(u + \Delta u, v)$, but this can be approximated by

$$g(u) + \frac{\partial g}{\partial u}(u, v)\Delta u$$

because the partial derivative of g with respect to u, at (u, v), is equal to the rate of change in the u-variable of g at (u, v). Therefore, the xy coordinates of $T(u + \Delta u, v)$ are approximately equal to

$$\left(g(u) + \frac{\partial g}{\partial u}(u, v)\Delta u, h(u) + \frac{\partial h}{\partial u}(u, v)\Delta u\right).$$

In a similar fashion, the xy coordinates of $T(u, v + \Delta v)$ are given by

$$\left(g(u) + \frac{\partial g}{\partial v}(u, v)\Delta v, h(u) + \frac{\partial h}{\partial v}(u, v)\Delta v\right).$$

We approximate the image of the small rectangle under T by the parallelogram which has these three vertices as adjacent vertices. The two sides of the parallelogram are represented by the vectors

$$\left\langle \frac{\partial g}{\partial u}(u,v)\Delta u, \frac{\partial h}{\partial u}(u,v)\Delta u \right\rangle, \left\langle \frac{\partial g}{\partial v}(u,v)\Delta v, \frac{\partial h}{\partial v}(u,v)\Delta v \right\rangle.$$

Recall that we have a method to calculate the area of a parallelogram spanned by two vectors! If we pretend these are two vectors in $\mathbb{R}^3$ by adding a z coordinate of 0, then the absolute value of their cross product is equal to the area of this parallelogram. The cross product of these two vectors (thought of as in $\mathbb{R}^3$ is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial g}{\partial u}(u,v)\Delta u & \frac{\partial h}{\partial u}(u,v)\Delta u & 0 \\ \frac{\partial g}{\partial v}(u,v)\Delta v & \frac{\partial h}{\partial v}(u,v)\Delta v & 0 \end{vmatrix} = \left(\frac{\partial g}{\partial u}(u,v)\Delta u \frac{\partial h}{\partial v}(u,v)\Delta v - \frac{\partial h}{\partial u}(u,v)\Delta u \frac{\partial g}{\partial v}(u,v)\Delta v \right) \mathbf{k}.$$

The end result of all of these approximations and calculations is that the image of the small $\Delta u \times \Delta v$ rectangle has area approximately equal to

$$\left| \frac{\partial g}{\partial u}(u,v)\Delta u \frac{\partial h}{\partial v}(u,v)\Delta v - \frac{\partial h}{\partial u}(u,v)\Delta u \frac{\partial g}{\partial v}(u,v)\Delta v \right| = \left| \frac{\partial g}{\partial u}(u,v) \frac{\partial h}{\partial v}(u,v) - \frac{\partial h}{\partial u}(u,v) \frac{\partial g}{\partial v}(u,v) \right| \Delta u \Delta v.$$

Therefore, if we use the images of all the small uv rectangles to form a Riemann sum in the xy plane, we end up with a sum of terms of the form

$$f(g(u^*,v^*), h(u^*,v^*)) \left| \frac{\partial g}{\partial u}(u,v) \frac{\partial h}{\partial v}(u,v) - \frac{\partial h}{\partial u}(u,v) \frac{\partial g}{\partial v}(u,v) \right| \Delta u \Delta v.$$

Define the function $J(u,v)$, called the *Jacobian* of T, to be

$$J(u,v) = \frac{\partial g}{\partial u}(u,v) \frac{\partial h}{\partial v}(u,v) - \frac{\partial h}{\partial u}(u,v) \frac{\partial g}{\partial v}(u,v).$$

The sums of these terms approximates the double integral

$$\iint_R f(x,y)\, dA,$$

on the one hand, since the sum is composed of terms over small pieces which make up R. On the other hand, this sum is also a Riemann sum for

$$\iint_S f(g(u,v), h(u,v)) |J(u,v)|\, du\, dv.$$

Therefore, it is reasonable to expect these two integrals to be equal to each other:

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v)) |J(u,v)|\, du\, dv.$$

Notice that this looks like a more general form of the various integration formulas we know for different coordinate systems: the integral of a function over a region R is expressed as an integral over a region S, which describes R using a different coordinate system, with an additional factor $J(u,v)$ inserted into the integrand.

We remark here that, at least in the two variable case, the Jacobian $J(u,v)$ is equal to the determinant of a 2×2 matrix:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{vmatrix}.$$

(The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $ad - bc$.)

Example. Let $x = r\cos\theta, y = r\sin\theta$ be the change of coordinates from (r, θ) to (x, y). What is the Jacobian of this transformation?

We begin by calculating the partial derivatives of x, y with respect to r, θ. We have

$$x_r = \cos\theta, x_\theta = -r\sin\theta, y_r = \sin\theta, y_\theta = r\cos\theta.$$

Therefore, the Jacobian is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

This explains the factor of r which appears in the integrand when switching to polar coordinates.

In the case where we use change of variables with three coordinates, from a uvw coordinate system to an xyz coordinate system, such as in cylindrical or spherical coordinates, it turns out that the Jacobian is given by the determinant of a 3×3 matrix whose entries are the various partial derivatives of x, y, z in terms of u, v, w. This is not an accident, because a fundamental property of determinants you learn in linear algebra is that the absolute value of the determinant of a matrix is equal to the volume of the parallelepiped spanned by its rows (or columns).

Example. Calculate the Jacobian of the transformation for rectangular coordinates; ie, the Jacobian of $x = r\cos\theta, y = r\sin\theta, z = z$.

The relevant partial derivatives are

$$x_r = \cos\theta, x_\theta = -r\sin\theta, x_z = 0, y_r = \sin\theta, y_\theta = r\cos\theta, y_z = 0, z_r = 0, z_\theta = 0, z_z = 1.$$

Therefore, the Jacobian of this transformation is given by the determinant

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

One can calculate this determinant using whatever formulas you may know; some knowledge of linear algebra tells us that this determinant is in fact equal to

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \times 1 = r.$$

(This is found by performing so-called *Laplace expansion* on the third row.)